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Q.No→ Define an 'analytic function'. Show that the function, $f(z) = e^{-z^{-4}}, (z \neq 0), f(0) = 0$ is analytic for all finite values of z except at $z = 0$ although the Cauchy-Riemann equations are satisfied at that point. How would you explain it.

Ans→ (Defn.) Analytic function:- A function $f(z)$ which is one valued and is differentiable at any point z_0 is said to be analytic at the point z.

If the function is analytic at every point z of a certain domain D, it is said to be analytic in that domain.

We have

$$f(z) = e^{-z^{-4}} = e^{-\frac{1}{z^4}} = e^{-\frac{1}{(x+iy)^4}}$$

$$= e^{-\frac{(x-iy)^4}{(x+iy)^4(x-iy)^4}}$$

$$= e^{-\frac{(x^4+y^4-6x^2y^2-4ix^3y+4ixy^3)}{(x^2+y^2)^4}}$$

$$= e^{-\frac{(x^4+y^4-6x^2y^2)}{r^8}} \cdot e^{\frac{4ixy(x^2-y^2)}{r^8}}$$

$[\because r^2 = x^2+y^2]$

$$= e^{-\frac{(x^4+y^4-6x^2y^2)}{r^8}} \times \left[\cos\left\{\frac{4xy(x^2+y^2)}{r^8}\right\} + i\sin\left\{\frac{4xy(x^2+y^2)}{r^8}\right\}\right]$$

Hence, at the origin,

$$\frac{\partial u}{\partial x} = \underset{x\to 0}{Lt}\ \frac{u(x,0)-u(0,0)}{x}$$

$$= \underset{x\to 0}{Lt}\ \frac{e^{-\frac{x^4}{x^8}} - 0}{x} = \underset{x\to 0}{Lt}\ \frac{e^{-1/x^4}}{x} = \underset{x\to 0}{Lt}\ \frac{1}{x e^{1/x^4}}$$

$$= \underset{x\to 0}{Lt}\ \frac{1}{x\left(1+\frac{1}{x^4}+\frac{1}{\underline{|2}}x^8+ \cdots\right)} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y\to 0}{Lt}\ \frac{u(0,y)-u(0,0)}{y}$$

$$= \underset{y\to 0}{Lt}\ \frac{e^{-\frac{y^4}{y^8}} - 0}{y} = \underset{y\to 0}{Lt}\ \frac{e^{-1/y^4}}{y}$$

$$= \underset{y\to 0}{Lt}\ \frac{1}{y e^{1/y^4}} = \underset{y\to 0}{Lt}\ \frac{1}{\left(1+\frac{1}{y^4}+\frac{1}{\underline{|2}}y^8 - \cdots\right)} = 0$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{Lt} \frac{v(x,0) - v(0,0)}{x} = \underset{x \to 0}{Lt} \frac{0-0}{x} = 0.$$

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{Lt} \frac{v(0,y) - v(0,0)}{y} = \underset{y \to 0}{Lt} \frac{0-0}{y} = 0.$$

Therefore, the Cauchy-Riemann equation are satisfied at the origin. Since $f(z)$ is infinite discontinuity at the origin at analytic.

M.U. MSC.98

QNo> If $f(z) = \frac{x^3 y(y-ix)}{x^6+y^2}$, $z \neq 0$ and $f(z)=0$.

Prove that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any maner

Soln We have

$$\underset{z \to 0}{Lt} \frac{f(z)-f(0)}{z} = \underset{z \to 0}{Lt} \frac{\frac{x^3 y(y-ix)}{x^6+y^2} - 0}{x+iy}$$

$$= \underset{z \to 0}{Lt} \frac{x^3 y(y-ix)}{(x^6+y^2)(x+iy)}$$

Let $z \to 0$ along any radius vector $y = mx$

$$= \underset{x \to 0}{Lt} \frac{x^3 mx(mx-ix)}{(x^6+m^2x^2)(x+imx)}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}\, m x \cdot x(m-i)}{\cancel{x^2}(x^4+m^2)\,\cancel{x}(1+im)}$$

$$= \lim_{x\to 0} \frac{m x^2(m-i)}{(m^2+x^4)(1+im)} = 0$$

Now, ~~let~~ It $z=0$, along $y=x^3$, then,

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{x^3 y(y-ix)}{(x^6+y^2)(x+iy)}$$

$$= \lim_{x\to 0} \frac{x^3 \cdot x^3(x^3-ix)}{(x^6+(x^3)^2)(x+ix^3)}$$

$$= \lim_{x\to 0} \frac{x^7(x^2-i)}{(x^6+x^6)(x+ix^3)}$$

$$= \lim_{x\to 0} \frac{\cancel{x^7}(x^2-i)}{2\cancel{x^6}(1+1)\cancel{x}(1+ix^2)}$$

$$= \lim_{x\to 0} \frac{(x^2-i)}{2(1+ix^2)}$$

$$= -\frac{i}{2}.$$

Hence the result.

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Q No.→ Examine the nature of the function,

$$f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}},\ z\neq 0,\ f(0)$$

in a region including the origin

Soln.- We have

$x^2y^5(x+iy)$

$$\underset{z\to 0}{Lt}\ \frac{f(z)-f(0)}{z} = \underset{z\to 0}{Lt}\ \frac{\left[\frac{x^2y^5(x+iy)}{x^4+y^{10}} - 0\right]}{(x+iy)}$$

$$= \underset{z\to 0}{Lt}\ \frac{x^2y^5}{x^4+y^{10}}$$

Let $z\to 0$, along $y = x$, then

$$= \underset{x\to 0}{Lt}\ \frac{x^2 . x^5}{x^4+x^{10}} = \underset{x\to 0}{Lt}\ \frac{\cancel{x^7}\, x^3}{\cancel{x^4}(1+x^6)} = \underset{x\to 0}{Lt}\ \frac{x^3}{1+x^6} = 0$$

Now, let $z\to 0$ along $y^5 = x^2$, then

$$\underset{x\to 0}{Lt}\ \frac{x^2 . x^2}{x^4+x^4} = \frac{\cancel{x^4}}{\cancel{x^4}(1+1)} = \frac{1}{2}.$$

Therefore, the Cauchy - Riemann are Satisfied, For we have

$$f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}} = \frac{x^3y^5}{x^4+y^{10}} + i\frac{x^2y^6}{x^4+y^{10}} = u+iv$$

$$\therefore u = \frac{x^3y^5}{x^4+y^{10}},\ v = \frac{x^2y^6}{x^4+y^{10}}$$

at the origin,

$$\frac{\partial u}{\partial x} = \underset{x\to 0}{Lt}\ \frac{u(x,0)-u(0,0)}{x} = \underset{x\to 0}{Lt}\ \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y\to 0}{Lt}\ \frac{u(0,y)-u(0,0)}{y} = \underset{y\to 0}{Lt}\ \frac{0-0}{y} = 0.$$

$$\frac{\partial v}{\partial x} = \underset{x\to 0}{Lt}\ \frac{v(x,0)-v(0,0)}{x} = \underset{x\to 0}{Lt}\ \frac{0-0}{x} = 0.$$

$$\frac{\partial v}{\partial y} = \underset{y\to 0}{Lt}\ \frac{v(0,y)-v(0,0)}{y} = \underset{y\to 0}{Lt}\ \frac{0-0}{y} = 0.$$

Q.No → Show that an analytic function with constant modulus is constant.

or, Q.No → Show that an analytic function cannot have a constant absolute value without reducing to a constant.

Solu^n. Let $f(z) = u + iv$.

Since, $|f(z)| = $ Constant, $= c$, say we have

$$u^2 + v^2 = c^2 \quad \text{--- (1)}$$

$\therefore 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$ and $2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$.

Using Cauchy-Riemann equ^n these become.

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0, \text{ and } u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0$$

Eliminating, $\frac{\partial u}{\partial y}$, we get $(u^2+v^2)\frac{\partial u}{\partial x} = 0$

Thus, $\frac{\partial u}{\partial x} = 0$ Provided $w = u + iv \neq 0$

Similarly, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$, & $\frac{\partial v}{\partial y} = 0$.

Since, the four partial derivatives of u; v are zero, the functions u, v are const. and consequently, $w = u + iv$ is also constt.

Q.No → If $f(z) = u + iv$ is an analytic function of $z = x + iy$, and ψ any function of x & y with differential coefficient of the first and second orders, then

(i) $\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 = \left\{\left(\frac{\partial \psi}{\partial u}\right)^2 + \left(\frac{\partial \psi}{\partial v}\right)^2\right\} |f'(z)|^2$

(ii) $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \left(\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right) |f'(z)|^2$

and (iii) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

Soln. We know that u & v are function of x & y, therefore, x & y will be expressed the function of u & v.

Hence, ψ will be a function u & v also therefore,

(i) $\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial\psi}{\partial v}\cdot\frac{\partial v}{\partial x}$ —— (1)

and $\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial\psi}{\partial v}\cdot\frac{\partial v}{\partial y}$

$= -\frac{\partial\psi}{\partial u}\cdot\frac{\partial v}{\partial x} + \frac{\partial\psi}{\partial v}\cdot\frac{\partial u}{\partial x}$ —— (2)

Squaring and adding (1) & (2), we get

$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = \left[\left(\frac{\partial\psi}{\partial u}\right)^2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\}\right]$

$+ \left[\left(\frac{\partial\psi}{\partial v}\right)^2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\}\right]$

$= \left[\left(\frac{\partial\psi}{\partial u}\right)^2 + \left(\frac{\partial\psi}{\partial v}\right)^2\right]\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$

$= \left[\left(\frac{\partial\psi}{\partial u}\right)^2 + \left(\frac{\partial\psi}{\partial v}\right)^2\right]|f'(z)|^2$ Proved

Since, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

(ii) Since $w = u + iv$ & $\bar{w} = u - iv$, we have

$\therefore u = \frac{1}{2}(w+\bar{w})$; $v = \frac{1}{2i}(w-\bar{w})$

$\therefore \frac{\partial}{\partial w} = \frac{\partial}{\partial u}\cdot\frac{\partial u}{\partial w} + \frac{\partial}{\partial v}\cdot\frac{\partial v}{\partial w}$

$= \frac{1}{2}\left(\frac{\partial}{\partial w} - i\frac{\partial}{\partial v}\right)$

$\therefore \frac{\partial}{\partial \bar{\omega}} = \frac{\partial}{\partial u} \frac{\partial u}{\partial \bar{\omega}} + \frac{\partial}{\partial v} \cdot \frac{\partial v}{\partial \bar{\omega}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v}\right)$

Multiplying these two integers, we have

$$\frac{\partial}{\partial \omega} \cdot \frac{\partial}{\partial \bar{\omega}} = \frac{1}{4}\left\{\left(\frac{\partial}{\partial u}\right)^2 + \left(\frac{\partial}{\partial v}\right)^2\right\}$$

$$4 \frac{\partial}{\partial \omega} \cdot \frac{\partial}{\partial \bar{\omega}} = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \therefore \frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} = 4 \frac{\partial \psi}{\partial \omega} \cdot \frac{\partial \psi}{\partial \bar{\omega}} \quad \text{--- (3)}$$

Similarly, it we can shown that,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 4 \frac{\partial \psi}{\partial z} \cdot \frac{\partial \psi}{\partial \bar{z}} \quad \text{--- (4)}$$

Since, $\omega = f(z)$, $\bar{\omega} = f(\bar{z})$.

$$\therefore \frac{\partial^2}{\partial \omega \, \partial \bar{\omega}} = \left(\frac{\partial}{\partial z} \cdot \frac{\partial z}{\partial \omega} \cdot \frac{\partial}{\partial \bar{z}} \cdot \frac{\partial \bar{z}}{\partial \bar{\omega}}\right)$$

$$= \left(\frac{1}{f'(z)} \cdot \frac{\partial}{\partial z}, \frac{1}{f'(\bar{z})} \frac{\partial}{\partial \bar{z}}\right) = \frac{1}{f'(z) f'(\bar{z})} \cdot \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}}$$

$$\therefore \frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} = 4 \frac{\partial^2 \psi}{\partial \omega \partial \bar{\omega}}$$

$$= 4 \frac{1}{|f'(z)|^2} \cdot \frac{\partial^2 \psi}{\partial z \cdot \partial \bar{z}}$$

$$= \frac{1}{|f'(z)|^2}\left\{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right\}$$

$$\therefore \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \left\{\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right\} |f'(z)|^2 \quad \text{Proved}$$

(iii) We have, Prove that

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 4 \frac{\partial^2 \psi}{\partial z \, \partial \bar{z}}$$

Replacing ψ by $|f(z)|^2$, we have

$$\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}|f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|f(z)|^2$$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\cdot f(z)\,f(\bar{z})$$

$$= 4\frac{\partial f(z)}{\partial z}\cdot\frac{\partial f(\bar{z})}{\partial \bar{z}}$$

$$= 4 f'(z)\cdot f'(\bar{z})$$

$$= 4|f'(z)|^2 \text{ Proved.}$$

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Q.No → If $w = f(z)$ is an analytic function of z such that $f'(z) \neq 0$, Prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$$

If $|f'(z)|$ is the Product of a function of x and a function of y Show that,

$$f'(z) = \exp(\alpha z^2 + \beta z + \gamma)$$

Where α is real and β and γ are complex constants.

Solu^n We have,

$$\log|f'(z)| = \frac{1}{2}\log|f'(z)|^2$$

$$= \frac{1}{2}\log[f'(z)\cdot f'(\bar{z})]$$

$$= \frac{1}{2}\log f'(z) + \frac{1}{2}\log f'(\bar{z})$$

We know that,

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$$

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[\frac{1}{2}\log f'(z)+\frac{1}{2}\log f'(\bar{z})\right]=0$$

Since z & $\bar{z}$ are independent this $=0$, we have

Let $|f'(z)| = \phi(x)\,\psi(y)$.

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left[\log\phi(x)+\log\psi(y)\right]=0$$

$$\therefore \frac{\partial^2}{\partial x^2}\log\phi(x) + \frac{\partial^2}{\partial y^2}\log\psi(y) = 0$$

Since, there is no relation x & y, we have

$$\frac{\partial^2}{\partial x^2} = \frac{d^2}{dx^2}\log\phi(x) = c$$

and $$\frac{\partial^2}{\partial y^2} = \frac{d^2}{dy^2}\log\psi(y) = -c.$$

Where, c is a real Constt.

From $\frac{d^2}{dx^2}\log\phi(x) = c$, we have

$$\log\phi(x) = \frac{cx^2}{2} + dx + e$$

$$\therefore \phi(x) = \exp\left(\frac{cx^2}{2} + dx + e\right)$$

Similarly, from $\frac{d^2}{dy^2}\log\psi(y) = -c$.

$$\log\psi(y) = -\frac{cy^2}{2} + d'y + e'$$

$$\psi(y) = \exp\left(-\frac{cy^2}{2} + d'y + e'\right)$$

Where, d', e, e' are real Constt.

$$\therefore |f'(z)| = \phi(x)\psi(y) = \exp\left[\frac{c}{2}(x^2-y^2)+dx+d'y+e+e'\right] \quad -(1)$$

$$|\exp(\alpha z^2+\beta z+\gamma)| = |\exp\{\alpha(x+iy)^2+(a+ib)(x+iy) + (c+di)\}|$$

Where, $\beta = a+ib$, & $\gamma = c+di$.

$= \exp\{\alpha(x^2 - y^2) + ax - by + c\}$ $[\because |e^{A+iB}| = e^A]$

which is of the same form as ①

Hence, we can write,

$$f'(z) = \exp(\alpha z^2 + \beta z + \gamma).$$